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Stability analysis and regularization of uncertain linear multi-objective integer optimization problems

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This article presents a systematic approach to analysing linear integer multi-objective optimization problems with uncertainty in the input data. The goal of this approach is to provide decision makers with meaningful information to facilitate the selection of a solution that meets performance expectations and is robust to input parameter uncertainty. Standard regularization techniques often deal with global stability concepts. The concept presented here is based on local quasi-stability and includes a local regularization technique that may be used to increase the robustness of any given efficient solutions or to transform efficient solutions that are not robust (*i.e.* unstable), into robust solutions. An application to a multi-objective problem drawn from the mining industry is also presented.

Keywords: stability analysis; post-optimality analysis; regularization; multi-objective optimization

1. Introduction

In many industries, the drive to improve the performance of process operations has lead to the integration of formerly independent operations. This integration promises significant increases in productivity, while reducing costs; however, it often leads to conflicting interests and the need for multi-objective optimization. For example, in the oil sands industry, the surface mining operation has become tightly coupled to the downstream extraction process. This coupling places a high demand on the trucking fleet continuously to deliver high-quality ore from the mine; however, this is in conflict with the reliability and maintenance cost containment goals for mining equipment.

In this type of environment, decisions must be made contingent upon uncertainties arising from many causes, such as: measurement errors, incomplete data, subjective information, fastchanging or time-varying process variables. Full understanding of the impact of these uncertainties on optimal operating decisions is critical to achieve an effective business. For this reason, tools are needed to understand the influence of uncertain input data and to mitigate the impact of such

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uncertainty on the solution. Specifically, decision makers must have answers to the following questions.

- (1) Is the selected efficient solution robust to uncertainty in the input data (*i.e.* will the efficient solution remain as such under variation in the input data)?
- (2) How robust is the efficient solution (*i.e.* how much uncertainty can be tolerated without losing efficiency)?
- (3) Which directions of change in the input data are safe (*i.e.* input data directions that preserve the efficiency of the solution)?
- (4) Which directions of change in the input data increase the robustness of the solution (*i.e.* could the efficient solution become more robust if the input data were refined)?
- (5) Which input parameters have the largest impact on the problem (*i.e.* where is the effort best spent in reducing input data uncertainty)?

Two approaches currently exist in the literature to deal with the effect of input uncertainty: the stochastic optimization approach and the post-optimality analysis method. In the stochastic optimization approach, uncertainty is quantified in a probabilistic manner yielding a stochastic optimization problem. Thus, the uncertainty is modelled *a priori*. Here, the post-optimality approach, which determines stability information after solving the optimization problem, is used. Some previously published articles based on this approach are Kozeratskaya *et al.* (1988, 1993, 2004), Emelichev and Podkopaev (1998) and Emelichev *et al.* (2002).

The aim of the present post-optimality analysis approach is to help decision makers understand the impact that uncertainty in the input data has on the efficient solution of integer problems. This method begins with the decision maker selecting one efficient solution from the Pareto set. The Pareto set represents the set of solutions where any improvement with respect to one of the objective functions will worsen at least one of the other objectives. (Different definitions and formulations of Pareto optimality are proposed in Jahn 2004 and Banke et al. 2008.) The selected solution is then analysed to determine if it is robust to uncertainty in the input data. If the solution is robust to uncertainty in the input data, it is classified as quasi-stable and the level of its robustness is quantified by the local quasi-stability radius. Information about the limiting level of changes that preserves the efficiency of the solution is derived from this local quasi-stability radius calculation. If the solution is not robust, it is classified as quasi-unstable and the local quasi-stability radius is zero. This means that some infinitesimal changes in the input data could cause the efficient solution to lose efficiency; however, in this case or the case of the stability radius being too small, the proposed method also provides directional information that can be used within the proposed regularization technique to increase the quasi-stability radius of the efficient solution. Therefore, if the decision maker can alter the input data or expects changes in it to occur in those directions, they can still implement the efficient solution knowing that it will become more robust in the future.

In contrast to the method proposed here, a global regularization technique that introduces specific perturbations of the original problem based on the notions of ordering and dual cones has been developed by several research groups (*e.g.* Kozeratskaya *et al.* 1988, 1993, 2004 and Emelichev *et al.* 2002). Since their approach is global, their interest lies in the stability of the entire Pareto set. Thus, the optimization problem is stable if all the efficient points are stable. Within this approach, to regularize the problem means to transform an unstable problem into a stable problem. Their definition of the global stability radius is the limiting level of changes to the input data that does not cause the appearance of new efficient solutions. Therefore, a stable problem has a global stability radius greater than zero and an unstable problem has a radius of zero. The latter means that some infinitesimal changes cause an inferior or weakly efficient point to become efficient. Their method requires the implementation of some specific perturbations of

the coefficients. Once this is done, the set of weakly efficient points of the transformed problem, which is part of the set of the efficient solutions of the original problem, becomes stable. Their technique, although mathematically interesting, does not adequately accommodate the decision making process for the following reasons.

- (a) The global stability radius quantifies the limiting level of changes to the input data that does not cause new efficient solutions to appear (*i.e.* it may not quantify the limiting level of changes that preserves the efficiency of the selected solution).
- (b) The stability of the problem is dependent on the input data transformations being realized (*i.e.* for the stability information to be useful, the decision maker must actually implement the deterministic changes, even if they may not be realistic).
- (c) Transforming the unstable problem into a stable one may leave out some of the efficient solutions of the original problem (*i.e.* a more preferred solution may exist in the original problem).
- (d) The global stability radius is conservative compared to a local stability radius (*i.e.* usually more uncertainty can be tolerated by a local solution).
- (e) The entire efficient set of the problem must be known prior to applying the regularization technique (*i.e.* this method is computationally difficult for large-scale problems).

Recently, a local quasi-stability concept has been introduced by Emelichev *et al.* (2010), where uncertainty lies in the criteria space C. Here, the proposed approach considers uncertainty from (1) the criteria space C and the constraint space (A, b) separately and (2) all the input data (C, A, b) at the same time. Therefore, this new post-optimality method is superior to previous attempts as it helps build the decision maker's insight as to the impact of uncertainty on the efficient solution and better reflects the actual decision making process. In summary, the main benefits of the present approach are as follows.

- (1) It determines whether the efficient solution is robust to uncertainty in the original input data.
- (2) It quantifies the limiting level of changes that preserves the efficiency of the selected solution.
- (3) The regularization is easily understood and requires little extra work once the quasi-stability radius is computed.
- (4) It indicates safe directions within the parameter space in which the problem may be varied (*i.e.* directions that preserve or increase the efficiency of the solution).
- (5) The decision maker has the opportunity to express preferences by choosing an efficient solution as a starting point of the analysis.

This article is organized as follows. First, the main concepts and definitions related to multiobjective optimization are presented briefly. Then solution stability is tackled and a new concept of local-stability radius is introduced; several definitions and results will be presented accordingly. Later, post-optimality analysis is discussed and a new regularization technique based on the local stability radius concept is presented. Finally, a case study drawn from the mining industry is used as an illustration.

2. Multi-objective optimization

The general form of a multi-objective integer linear optimization problem with n decision variables, m constraints and L objectives is as follows:

$$\max_{x} Cx$$

s.t. $Ax \leq b$,
 $x \in \mathbb{Z}^{n}$, (1)

where $X \stackrel{\text{def}}{=} \{ x \in \mathbb{Z}^n : Ax \leq b \}$ represents the feasible set and $C \stackrel{\text{def}}{=} [c_{ij}] \in \mathbb{R}^{L \times n}$ is the coefficient matrix whose rows are the gradients $c_i, i = 1, ..., L$, of the *i*th objective function.

A key difference between single and multi-objective optimization problems is the ordering of the feasible domain with respect to the objective function(s). In single-objective optimization problems, full ordering of the feasible domain with respect to the objective function is possible, which makes identification of an optimal solution straightforward. In multi-objective optimization problems, with opposing objectives, the ordering is not as simple as in the single-objective case, since a change that represents an improvement with respect to one of the objective functions will normally worsen at least one of the other objectives. Thus, optimality is replaced by the notion of Pareto optimality, or more precisely, the notion of efficiency. Along with the notion of efficiency, two other variants exist (see Sawaragi *et al.* 1985): weak efficiency and strict efficiency.

DEFINITION 2.1 Inferior point. A feasible point $x_0 \in X$ is inferior if it is dominated by another feasible point, i.e.

$$\exists x \in X, \quad \forall i = 1, \ldots, L, \quad c_i x > c_i x_0.$$

DEFINITION 2.2 Weakly efficient point. A feasible point $x_0 \in X$ is weakly efficient if it is not inferior. P(C, X) denotes the set of all weakly efficient points, i.e.

$$\boldsymbol{P}(\boldsymbol{C},\boldsymbol{X}) \stackrel{\text{def}}{=} \{\boldsymbol{x}_0 \in \boldsymbol{X} : \forall \boldsymbol{x} \neq \boldsymbol{x}_0 \in \boldsymbol{X}, \exists i_0 \in \{1,\ldots,L\} \ \boldsymbol{c}_{i_0} \boldsymbol{x} \leq \boldsymbol{c}_{i_0} \boldsymbol{x}_0\}.$$

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Weak efficiency is the most relaxed form of efficiency. Weakly efficient points provide the decision maker with conflicting information. In the literature other efficiency concepts are introduced in Sawaragi *et al.* (1985) and Jahn (2004).

DEFINITION 2.3 Efficient point. An efficient point $x_0 \in X$ is a weakly efficient point with at least one objective not being dominated. $\Pi(C, X)$ denotes the set of all efficient points, i.e.

$$\mathbf{\Pi}(\boldsymbol{C},\boldsymbol{X}) \stackrel{\text{def}}{=} \{\boldsymbol{x}_0 \in \boldsymbol{P}(\boldsymbol{C},\boldsymbol{X}): \ \nexists \boldsymbol{x} \neq \boldsymbol{x}_0 \in \boldsymbol{X}, \ \exists j_0 \in \{1,\ldots,L\}, \ \boldsymbol{c}_{j_0}\boldsymbol{x} \geq \boldsymbol{c}_{j_0}\boldsymbol{x}_0\}.$$

DEFINITION 2.4 Strictly efficient point. A strictly efficient point is a weakly efficient point with at least one objective dominating. S(C, X) denotes the set of all strictly efficient points, i.e.

$$S(C, X) \stackrel{\text{def}}{=} \{ x_0 \in P(C, X) : \exists k_0 \in \{1, \dots, L\}, \ c_{k_0} x < c_{k_0} x_0 \}.$$

Strict efficiency is the most restrictive form of efficiency. The set of strictly efficient points is included in the set of efficient points, which is included in the set of weakly efficient points:

$$S(C,X) \subset \Pi(C,X) \subset P(C,X).$$

To illustrate these ideas, consider the following multi-objective optimization problem:

$$\max_{\mathbf{x}} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$$

s.t.
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} \le \begin{bmatrix} 6 \\ 6 \\ 0 \\ 0 \end{bmatrix}.$$
 (2)

The corresponding feasible domain and the sets of weakly efficient, efficient and strictly efficient points are illustrated in Figure 1 for both the decision space and the objective space. Since both



Figure 1. Example describing different types of efficiency: (a) decision space, (b) objective space.



Figure 2. Example describing different types of efficiency in the decision space when all the variables are discrete.

objectives are linear and the decision variable x is continuous, the efficient solutions lie on the boundary of the feasible domain. For instance, the points lying on the line segments AB and BC are weakly efficient. Moreover, the line segment BC represents the subset of points that are also strictly efficient. This is clearly shown in Figure 1(b) where improving the second objective along AB does not impact the first objective and where improving either objective along BC worsens the other objective. All the other feasible points, excluding those lying on the line segments AB or BC are inferior. Point D is an example of an inferior point.

In the same example, but defining $x \in \mathbb{Z}^n$, in Figure 2 the set of weakly efficient points is $\{A, B, C, E\}$ and the set of strictly efficient points is $\{B, C, E\}$. The point D is an example of an inferior point, since it is dominated by point B.

3. Solution stability

Since most practical problems have uncertain input data, a stability analysis of the problem is a very important part of the decision making process. Therefore, quantifying the stability (robustness) of an efficient solution with respect to changes in the input parameters is highly desirable. For



Figure 3. Quasi-stability with respect to matrix C in the decision space.



Figure 4. Quasi-stability with respect to the constraints in which: (a) matrix A is perturbed, and (b) vector b is perturbed.

example, it is valuable to know that slightly varying a parameter in one direction may result in the efficient point becoming inferior in the perturbed problem. As well, it is of equal value to know that greatly varying a parameter in another direction may result in preserving the efficiency of the point.

The idea of stability is illustrated in Figures 3 and 4 for the example given in problem (2). For instance, in Figure 3, point E is strictly efficient and can withstand slight perturbations up to the first objective passing through C. Therefore E is quasi-stable with respect to the criterion matrix C and has a quasi-stability radius equal to that amount of change. At this limit, E loses strict efficiency and becomes weakly efficient instead. If the first objective continues to change past this limit, E would become inferior to C. A similar analysis can be done for the other objective. Figures 4(a) and 4(b) illustrate that point E is also quasi-stable with respect to matrix A and vector b, respectively. This is because the constraint given by the line BC can be rotated and/or translated outwardly or inwardly slightly before point F becomes either feasible and dominant or before point E loses feasibility. The quasi-stability radius is again equal to the amount of change in matrix A or vector b.

For the general linear, integer, multi-objective optimization problem, in order to investigate stability with respect to perturbations/uncertainty in the input data coefficients C, A and b, the

perturbed problem has the form:

$$\max_{\mathbf{x}} C(\delta)\mathbf{x}$$

s.t. $A(\delta)\mathbf{x} \le b(\delta),$ (3)
 $\mathbf{x} \in \mathbb{Z}^{n}.$

For the sake of simplicity, only additive perturbations are considered: $C(\delta) \stackrel{\text{def}}{=} C + C', A(\delta) \stackrel{\text{def}}{=} A + A'$ and $b(\delta) \stackrel{\text{def}}{=} b + b'$, where C', A' and b' are the matrices and the vector of perturbations. More general perturbations could be considered by making use of norms in function spaces to deal with the local quasi-stability radius.

The literature on stability and regulation of multi-objective integer linear programs often uses a global quasi-stability concept (Emelichev *et al.* 2002). In that case the stability is studied with respect to all the strictly efficient solutions of the problem and the following global definition must be used.

DEFINITION 3.1 Global quasi-stability. The original integer problem (C, A, b) is quasi-stable if $\exists \delta_0 > 0$, such that $\forall \delta \in (0, \delta_0)$, $\forall C' : \|C'\|_p < \delta, \forall A' : \|A'\|_p < \delta, \forall b' : \|b'\|_p < \delta$ then

$$S(C,X) \subset S(C+C',X+X')$$

where $1 \le p \le +\infty$ and $X + X' \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^n | (A + A')x \le b + b'\}.$

In practice, once a solution is given, the goal is to investigate its robustness with respect to the perturbations. Therefore, a local stability concept specific to each solution is introduced. The proposed definition is the following.

DEFINITION 3.2 Local quasi-stability. A strictly efficient point \mathbf{x}_0 of the problem $(\mathbf{C}, \mathbf{A}, \mathbf{b})$ is quasistable if $\exists \delta_0 > 0$, such that $\forall \delta \in (0, \delta_0)$, $\forall \mathbf{C}' : \|\mathbf{C}'\|_p < \delta, \forall \mathbf{A}' : \|\mathbf{A}'\|_p < \delta, \forall \mathbf{b}' : \|\mathbf{b}'\|_p < \delta$ the point is also strictly efficient for the perturbed problem $(\mathbf{C} + \mathbf{C}', \mathbf{A} + \mathbf{A}', \mathbf{b} + \mathbf{b}')$.

To illustrate the concept of stability, the uncertainty is assumed to lie only in the input matrix C and the following problem is considered:

$$\max_{x} C(\delta)x$$
s.t. $Ax \leq b$, (4)
 $x \in \mathbb{Z}^{n}$.

In this case, the distance between problem (1) and (4) is defined as the distance between the original matrix C and the perturbed one $C(\delta)$. Now, the aim is to measure this distance such that the efficiency of a specific solution will be preserved. In other words, the limiting level of perturbations, such that the efficiency of the solution will be maintained, has to be quantified.

If x_0 is a strictly efficient solution, then this means that at least in one direction x_0 cannot be improved. Therefore, there exists an i_0 such that $c_{i_0}(x - x_0) < 0$, where x is an alternative feasible point. If efficiency is to be preserved, then the same property would have to be satisfied by the perturbed problem, namely $c_{i_0}(\delta)(x - x_0) < 0$. Therefore, the stability region of x_0 with respect to objective i_0 is the set defined as follows:

$$\mathcal{S}_{i_0}(\boldsymbol{x}_0) \stackrel{\text{def}}{=} \left\{ \delta > 0, \ \forall \boldsymbol{x} \in \boldsymbol{X}, \ \left\{ \begin{array}{l} \boldsymbol{c}_{i_0}(\delta)(\boldsymbol{x} - \boldsymbol{x}_0) < 0, \\ \boldsymbol{c}_{i_0}(\boldsymbol{x} - \boldsymbol{x}_0) < 0. \end{array} \right\}.$$
(5)

Outside this stability region, x dominates x_0 in direction i_0 .

Since $c_i(\delta) = c_i + c'_i$, if the stability region on a plot with the axes of coordinates c'_{ij} were represented, a stability region of a very irregular shape would be obtained. To make this stability information usable in practice, a way of measuring/approximating this stability region must be found, even though it may lead to losing information. This measure is given by the quasi-stability radius. The exact way this is done is presented in the next subsection.

3.1. Local quasi-stability radius

The quasi-stability radius quantifies the limiting level of perturbations of the coefficients C, A and b such that the efficiency of the solution is preserved. The formal definition is given below.

DEFINITION 3.3 Local quasi-stability radius. The local quasi-stability radius of a strictly efficient solution \mathbf{x}_0 of the problem (1) is the minimum of the distances from this problem to problem (4) for which \mathbf{x}_0 is no longer strictly efficient:

$$R^{q}(\boldsymbol{C},\boldsymbol{x}_{0}) \stackrel{\text{def}}{=} \min_{\delta>0} \quad \text{dist}((1),(4)) \quad such \ that \quad \boldsymbol{x}_{0} \notin \boldsymbol{S}(\boldsymbol{C}(\delta),\boldsymbol{X})$$
$$\stackrel{\text{def}}{=} \max_{1 \leq i \leq L} \|\boldsymbol{c}_{i}(\delta) - \boldsymbol{c}_{i}\|_{p},$$
$$\text{s.t.} \ \forall i = 1, \dots, L, \quad \exists \boldsymbol{x} \neq \boldsymbol{x}_{0} \in \boldsymbol{X}, \quad \boldsymbol{c}_{i}(\delta)(\boldsymbol{x} - \boldsymbol{x}_{0}) \geq 0, \tag{6}$$

where $\|\cdot\|_p$ stands for the *p*-norm on \mathbb{R}^n such that $1 \le p \le +\infty$ and $p^{-1} + q^{-1} = 1$.

The above definition of local quasi-stability radius has a global equivalent.

DEFINITION 3.4 Global quasi-stability radius. The global quasi-stability radius of the original problem is the minimum of the distances between the original and the perturbed problem that preserves the entire strictly efficient set.

In the same fashion, the perturbed problem and the corresponding definitions can be modified for cases where the uncertainty lies only in A or b, or any combination thereof.

3.2. Local quasi-stability radius (input matrix C)

The goal of this section is to provide the decision maker with stability information about the selected efficient solution with respect to uncertainty in the input matrix C. This stability information can be captured by the local quasi-stability radius R^q (C, x_0) (see Definition 3.3). If only perturbations with respect to the cost matrix are considered, the distance between problem (1) and the perturbed one (4) can be defined as: $\max_{1 \le i \le L} \|c_i(\delta) - c_i\|_p$.

The following theorem gives the formula of the local quasi-stability radius with respect to matrix C for an efficient solution x_0 .

THEOREM 3.5 Let \mathbf{x}_0 be a strictly efficient solution of problem (1) and $1 \le p \le +\infty$. Then, if only additive perturbations with respect to the cost matrix are considered, the quasi-stability radius on x_0 is given by

$$R^{q}(\boldsymbol{C}, \boldsymbol{x}_{0}) = \min_{\substack{x \neq x_{0} \\ A\boldsymbol{x} < \boldsymbol{b} \ c_{i}(\boldsymbol{x} - \boldsymbol{x}_{0}) < 0}} \max_{\substack{1 \le i \le L \\ (\boldsymbol{x} - \boldsymbol{x}_{0}) < 0}} \frac{|\boldsymbol{c}_{i}(\boldsymbol{x} - \boldsymbol{x}_{0})|}{\|\boldsymbol{x} - \boldsymbol{x}_{0}\|_{q}},\tag{7}$$

where $p^{-1} + q^{-1} = 1$.

Proof A variant of this proof can be found in Emelichev and Podkopaev (1998), which focuses on the global quasi-stability radius. The proof is focused on the local quasi-stability radius and is generalized with respect to the choice of the norms.

Let x_0 be a strictly efficient solution of the original problem (1). According to Definition 2.4:

$$\forall x \neq x_0 \in X, \quad \exists i_0 = 1, \dots, L, \quad c_{i_0}(x - x_0) < 0.$$
 (8)

Looking for the local quasi-stability radius on $x_0 R^q$ (C, x_0), to minimize the distance between problem (1) and problem (4) such that x_0 is not a strictly efficient point for the perturbed problem:

$$R^{q}(\boldsymbol{C},\boldsymbol{x}_{0}) \stackrel{\text{def}}{=} \min_{\boldsymbol{\delta}>0} \quad \text{dist}((1),(4)) \quad \text{such that} \quad \boldsymbol{x}_{0} \notin \boldsymbol{S}(\boldsymbol{C}(\boldsymbol{\delta}),\boldsymbol{X})$$
$$\stackrel{\text{def}}{=} \max_{1 \le i \le L} \|\boldsymbol{c}_{i}(\boldsymbol{\delta}) - \boldsymbol{c}_{i}\|_{p},$$
$$\text{s.t.} \exists \boldsymbol{x} \neq \boldsymbol{x}_{0} \in \boldsymbol{X}, \quad \forall i = 1, \dots, L, \quad \boldsymbol{c}_{i}(\boldsymbol{\delta})(\boldsymbol{x} - \boldsymbol{x}_{0}) \ge 0.$$
(9)

Looking only at additive perturbations, $c_i(\delta) = c_i + c'_i$ and the main objective of the optimization problem (9) can be stated as follows:

dist((1), (4))
$$\stackrel{\text{def}}{=} \max_{1 \le i \le L} \| \boldsymbol{c}_i(\delta) - \boldsymbol{c}_i \|_p = \max_{1 \le i \le L} \| \boldsymbol{c}'_i \|_p$$

where $\delta \stackrel{\text{def}}{=} \max_{1 \le i \le L} \|c'_i\|_p$. A real number δ satisfying the constraints of problem (9) will be such that $\exists x \ne x_0, \forall i = 1, ..., L$, such that

$$(c_i + c'_i)(x - x_0) \ge 0$$

$$\implies c'_i(x - x_0) \ge -c_i(x - x_0)$$

$$\implies |c'_i(x - x_0)| \ge -c_i(x - x_0).$$

By the Hölder inequality, $(|\mathbf{h}(\mathbf{x} - \mathbf{x}_0)| \le \|\mathbf{h}\|_p \|\mathbf{x} - \mathbf{x}_0\|_q$, where p, q > 0 and $p^{-1} + q^{-1} = 1$ (p, q are complementary to each other and $\|\cdot\|_p$ stands for the *p*-norm)):

$$\|\boldsymbol{c}_i'\|_p \|\boldsymbol{x} - \boldsymbol{x}_0\|_q \ge -\boldsymbol{c}_i(\boldsymbol{x} - \boldsymbol{x}_0) \\ \implies \quad \|\boldsymbol{c}_i'\|_p \ge -\frac{\boldsymbol{c}_i(\boldsymbol{x} - \boldsymbol{x}_0)}{\|\boldsymbol{x} - \boldsymbol{x}_0\|_q}$$

passing to the maximum through all i = 1, ..., L:

$$\max_{1\leq i\leq L} \|\boldsymbol{c}_i'\|_p \geq \max_{1\leq i\leq L} -\frac{\boldsymbol{c}_i(\boldsymbol{x}-\boldsymbol{x}_0)}{\|\boldsymbol{x}-\boldsymbol{x}_0\|_q}.$$

As there exists an i_0 such that $c_{i_0}(x - x_0) < 0$, then the maximum over all i = 1, ..., L such that $c_i(x - x_0) < 0$ (the focus is only on preserving efficiency with respect to at least one objective) can be considered. Then:

$$\delta = \max_{1 \le i \le L} \|\boldsymbol{c}_i'\|_p \ge \max_{\substack{1 \le i \le L \\ \boldsymbol{c}_i(\boldsymbol{x} - \boldsymbol{x}_0) < 0}} \frac{|\boldsymbol{c}_i(\boldsymbol{x} - \boldsymbol{x}_0)|}{\|\boldsymbol{x} - \boldsymbol{x}_0\|_q} > 0.$$

This property is available for all feasible points $x \neq x_0 \in X$. Thus,

$$\delta \geq \min_{\substack{\substack{x\neq x_0\\A\mathbf{x} \leq \mathbf{b}}}} \max_{\substack{1 \leq i \leq L\\c_i(\mathbf{x} - \mathbf{x}_0) < 0}} \frac{|\boldsymbol{c}_i(\mathbf{x} - \mathbf{x}_0)|}{\|\mathbf{x} - \mathbf{x}_0\|_q} > 0,$$

 δ is bounded from below and reaches its boundary (as it is positive). Definition 3.3 of the local quasi-stability radius conclude the proof.

Table 1. The stability radius and allowed perturbations (lightly shaded area – stability region; darkly shaded area – approximation of the stability region determined with the help of the stability radius).

 $\begin{aligned} \|\boldsymbol{x} - \boldsymbol{x}_0\|_q &= \max_{k=1,\dots,n} |x_k - x_{0k}| \\ p &= 1 \quad \sum_{k=1}^n |c_{ik}(\delta) - c_{ik}| < R^q(\boldsymbol{C}, \boldsymbol{x}_0) \\ q &= \infty \end{aligned}$ The entries of every row $(i = 1, \dots, L)$ can be perturbed such that the sum of the absolute values of the perturbations in every row are less than the quasi-stability radius.

 $\begin{aligned} \| \boldsymbol{x} - \boldsymbol{x}_0 \|_q &= \sum_{k=1}^n | \boldsymbol{x}_k - \boldsymbol{x}_{0k} | \\ p &= \infty \quad \max_{k=1,\dots,n} |c_{ik}(\delta) - c_{ik}| < R^q(\boldsymbol{C}, \boldsymbol{x}_0) \end{aligned}$

q = 1 The entries of every row (i = 1, ..., L) can be perturbed such that $|c_{ik}(\delta) - c_{ik}| < R^q$. The absolute value of the perturbations of every element of the matrix *C* are less than the quasi-stability radius.



 $p = 2 \qquad (\sum_{k=1}^{n} |\mathbf{c}_{k} - \mathbf{x}_{0k}|^{2})^{1/2} < R^{q}(\mathbf{C}, \mathbf{x}_{0})$

q = 2 The entries of every row (i = 1, ..., L) can be perturbed such that the square root of the sum of the squares of the perturbations in every row is less than the quasi-stability radius.

At the beginning of this section, it was stated that the stability space of an efficient solution has a very irregular shape and that an easily computable measure must be found. Depending on the choice of the *p*-norms, the quasi-stability radius provides alternative approximations to the quasi-stability region. Therefore, the quasi-stability radius can be considered to be a form of the required measure. The different approximations of the quasi-stability space corresponding to three choices of norms and the stability radii calculations based on these norms are summarized in Table 1.

Using norms to specify the approximation to the quasi-stability space has two important consequences. First, the norm definitions impact the information provided by the calculation to the decision maker. For instance, in Figure 5(a) each norm definition identifies the same limiting perturbation, but in Figure 5(b) the limiting perturbation is a function of the norm used. Secondly, the quasi-stability radius only quantifies the true quasi-stability of the efficient solution in exactly one direction. In every other direction the limiting perturbations are either exactly equal to or greater



Figure 5. The impact of the choice of norms used for the computation of the quasi-stability radius on the information provided to the decision maker.



Figure 6. Increasing the quasi-stability radius from 1 to 2 by strategically modifying the input parameter c_{21} . (a) The sensitive direction. (b) Modifying the coefficient in the opposite direction.

than the quasi-stability radius. Thus, opportunities exist to increase the quasi-stability radius by strategically modifying the input matrix C as illustrated in Figure 6.

SUMMARY 3.6 The information provided by the quasi-stability radius with respect to the matrix C can be summarized as follows.

- (1) If \mathbf{x}_0 is a strictly efficient solution and the perturbations to the input matrix $\mathbf{C} = [\mathbf{c}_i]$ are less than the quasi-stability radius for those rows i for which $\mathbf{c}_i(\mathbf{x} \mathbf{x}_0) \le 0$, \mathbf{x}_0 remains strictly efficient for the perturbed problem; the entries of the other rows are free, in the sense that they can be perturbed indefinitely.
- (2) If all the objective vectors \mathbf{c}_i for which $\mathbf{c}_i(\mathbf{x} \mathbf{x}_0) \leq 0$ are perturbed slightly beyond the quasi-stability radius, \mathbf{x}_0 becomes inferior for the perturbed problem.

Remark 1 The stability radius formula given by (7) can be easily generalized (see Appendix A).

Remark 2 Similar local quasi-stability radii can be computed when perturbations with respect to the input matrix A and vector b are used (see Appendix B) and with respect all entries C, A and b (see Appendix C).

Remark 3 In this article, the terms local quasi-stability radius and quasi-stability radius with respect to the efficient point x_0 have the same meaning.

3.3. Scaling issues

In the previous definitions, the quasi-stability radius quantified the absolute magnitude of the limiting level of perturbations such that the efficiency and feasibility of the solution are preserved. In practice, this can become problematic for two main reasons:

• parameters may have different units or vastly different orders of magnitude (*i.e.* the quasistability radius relative to each parameter is very different), • some parameters may have higher uncertainty or change more quickly than others (*i.e.* decision makers want to select solutions that are more robust with respect to these parameters).

For instance, to deal with these practical issues, the objective vector $\mathbf{c}_i = (c_{i1}, \ldots, c_{in})$ can be scaled by \tilde{c}_i , where \tilde{c}_i can be the 1-norm, the 2-norm, the ∞ -norm, the average of \mathbf{c}_i or any other positive scaling factor. The vector $\bar{\mathbf{c}}_i \stackrel{\text{def}}{=} \mathbf{c}_i/\tilde{c}_i$ must then replace \mathbf{c}_i in formula (7). After computing the quasi-stability radius, it is useful to return to the original variables. If the scaled quasi-stability radius is $\bar{R}_s^q(\mathbf{C}, \mathbf{x}_0) \stackrel{\text{def}}{=} R^q(\mathbf{C}, \mathbf{x}_0)|_{\mathbf{c}_i = \bar{c}_i}$, then the dimensioned quasi-stability radii are $(R_s^q)^i(\mathbf{C}, \mathbf{x}_0) \stackrel{\text{def}}{=} \bar{R}_s^q(\mathbf{C}, \mathbf{x}_0)|_{\mathbf{c}_i = c_i}$, for $i = 1, \ldots, L$. This means that the allowable perturbations, which preserve the efficiency of the original efficient point \mathbf{x}_0 , are $\|\mathbf{c}_i - \mathbf{c}_i(\delta)\|_p < (R_s^q)^i$.

It is also important to incorporate the impact of the uncertainty in the calculations. For instance, to emphasize the importance of an uncertain objective/constraint, that objective/constraint in the stability radius calculation must be weighted by a factor less than one. For example, to emphasize one objective, denoted c_l , since its rate of change is w_l times slower than the rate of the change of the other objectives, then the objective c_l must be replaced by $\bar{c}_l = c_l w_l$. This can also be done with constraints and has similar consequences as scaling. Notice that this type of transformation does not modify the optimization problem (1), but does alter the dimensionless quasi-stability radius to encourage the selection of a solution that is more stable with respect to the emphasized objective/constraint. These issues will be investigated in an upcoming article (Seck *et al.* 2012).

4. Local quasi-stability radius and post-optimal analysis

4.1. Sensitive directions

So far, the quasi-stability radius has only been used to quantify the limiting level of perturbations such that the efficiency and feasibility of the solution are preserved; however, even more information can be extracted from the quasi-stability radius calculation. Specifically, the calculation provides directional information that can be used to strategically perturb the input data coefficients C, A and b to increase the quasi-stability radius.

In this section, all the perturbations are assumed to be additive with respect to the cost matrix. Therefore, the perturbed problem (4) is considered.

Recall from Section 3.2 that a direct consequence of the norm definitions is that the local quasi-stability radius only quantifies the true quasi-stability of a strictly efficient solution x_0 in exactly one direction. Imagine that the approximation of the stability region, computed using the quasi-stability radius, touches the boundary of the original, irregular stability region at the point \hat{x} (*i.e.* \hat{x} is the attained minimum value of problem 9):

$$\hat{\boldsymbol{x}} \in \arg\min_{\delta>0} \quad \max_{1 \le i \le L} \|\boldsymbol{c}_i(\delta) - \boldsymbol{c}_i\|_p,$$

s.t. $\exists \boldsymbol{x} \neq \boldsymbol{x}_0, \ \forall i = 1, \dots, L, \ \boldsymbol{c}_i(\delta)(\boldsymbol{x} - \boldsymbol{x}_0) \ge 0.$ (10)

This point of contact is the optimal solution of the target quasi-stability radius equation. Assume that the optimal value is reached for $i_0 = l$. This means that the scaled quasi-stability radius is

$$R_{s}^{q}(\boldsymbol{C}, \boldsymbol{x}_{0}) = \frac{|\boldsymbol{c}_{l}(\hat{\boldsymbol{x}} - \boldsymbol{x}_{0})|}{\|\hat{\boldsymbol{x}} - \boldsymbol{x}_{0}\|_{q}}.$$
(11)

The following notations are used:

$$(R_{\rm s}^q)_i^l \stackrel{\text{def}}{=} \frac{|\boldsymbol{c}_i(\hat{\boldsymbol{x}} - \boldsymbol{x}_0)|}{\|\hat{\boldsymbol{x}} - \boldsymbol{x}_0\|_q}, \quad i = 1, \dots, L, \text{ if } \boldsymbol{c}_i(\hat{\boldsymbol{x}} - \boldsymbol{x}_0) < 0, \tag{12}$$

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$$\boldsymbol{d} \stackrel{\text{def}}{=} \operatorname{sign}(\hat{\boldsymbol{x}} - \boldsymbol{x}_0). \tag{13}$$

Note that $(R_{s}^{q})_{l}^{l} = R_{s}^{q}(C, x_{0}).$

The vector d provides the sensitive direction when dealing with perturbations with respect to the input matrix C.

Remark 4 If stability with respect to *A* and *b* is considered, the sign of the vector *d* is changed; therefore, in what follows, $d \stackrel{\text{def}}{=} \operatorname{sign}(x_0 - \hat{x})$.

As stated previously, the sensitive direction is the most important direction, since the quasistability radius only quantifies the true quasi-stability of an efficient solution x_0 in this direction. With this in mind, minimum perturbations with respect to C are presented so that a progression from the strictly efficient case to the inferior case is obtained. This is illustrated by the following propositions.

PROPOSITION 4.1 Assume that all the objective vectors c_i for which $c_i(\hat{x} - x_0) \le 0$ are perturbed and the perturbations are less (with respect to the p-norm) than the quasi-stability radius. Then x_0 remains strictly efficient for the perturbed problem.

Proof All the objective vectors c_i for which $c_i(\hat{x} - x_0) \le 0$ are perturbed and the perturbations are less than the quasi-stability radius. This means that $||c_i(\delta) - c_i||_p < R^q(C, x_0)$ for all i = 1, ..., L such that $c_i(\hat{x} - x_0) \le 0$

$$\|\boldsymbol{c}_i(\delta) - \boldsymbol{c}_i\|_p < R^q(\boldsymbol{C}, \boldsymbol{x}_0) \stackrel{\text{def}}{=} (R^q_{\text{s}})_l^l(\boldsymbol{C}, \boldsymbol{x}_0) = \frac{|\boldsymbol{c}_l(\hat{\boldsymbol{x}} - \boldsymbol{x}_0)|}{\|\hat{\boldsymbol{x}} - \boldsymbol{x}_0\|_q}.$$

By (10), $\exists x \neq x_0$,

$$\begin{aligned} \|\boldsymbol{c}_{i}(\delta) - \boldsymbol{c}_{i}\|_{p} < R^{q}(\boldsymbol{C}, \boldsymbol{x}_{0}) &= (R_{s}^{q})_{l}^{l}(\boldsymbol{C}, \boldsymbol{x}_{0}) = \frac{|\boldsymbol{c}_{l}(\boldsymbol{x} - \boldsymbol{x}_{0})|}{\|\boldsymbol{\hat{x}} - \boldsymbol{x}_{0}\|_{q}} \leq \frac{|\boldsymbol{c}_{l}(\boldsymbol{x} - \boldsymbol{x}_{0})|}{\|\boldsymbol{x} - \boldsymbol{x}_{0}\|_{q}}, \\ \implies \|\boldsymbol{c}_{i}(\delta) - \boldsymbol{c}_{i}\|_{p}\|\boldsymbol{x} - \boldsymbol{x}_{0}\|_{q} < |\boldsymbol{c}_{l}(\boldsymbol{x} - \boldsymbol{x}_{0})|, \\ \implies |(\boldsymbol{c}_{i}(\delta) - \boldsymbol{c}_{i})(\boldsymbol{x} - \boldsymbol{x}_{0})| < |\boldsymbol{c}_{l}(\boldsymbol{x} - \boldsymbol{x}_{0})| \quad \text{(by the Hölder inequality).} \end{aligned}$$

This is true for all $i = 1, \ldots, L$. Then:

$$\left| c_l(\delta)(\mathbf{x} - \mathbf{x}_0) - \underbrace{c_l(\mathbf{x} - \mathbf{x}_0)}_{<0} \right| < |c_l(\mathbf{x} - \mathbf{x}_0)|.$$

Then, by the strict efficiency of x_0 with respect to the original problem:

$$\boldsymbol{c}_l(\delta)(\boldsymbol{x}-\boldsymbol{x}_0) < 0.$$

Therefore x_0 is strictly efficient for the perturbed problem.

PROPOSITION 4.2 Assume that all the objective vectors \mathbf{c}_i are perturbed in the sensitive direction and $\mathbf{c}_i(\hat{\mathbf{x}} - \mathbf{x}_0) \leq 0$ for all i = 1, ..., L. Also assume that the perturbations are equal to the quasistability radius. Then both \mathbf{x}_0 and $\hat{\mathbf{x}}$ become efficient but not strictly efficient for the perturbed problem.

Proof (1) To prove that x_0 is efficient for the perturbed problem, proof by contradiction is used.

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Assume that $\exists \mathbf{x}' \neq \mathbf{x}_0 \in \mathbf{X}$ such that $c_l(\delta)(\mathbf{x}' - \mathbf{x}_0) \ge 0$. Then, by the fact that the perturbations are equal to the quasi-stability radius and by (10):

$$\begin{aligned} \|\boldsymbol{c}_{l}(\delta) - \boldsymbol{c}_{l}\|_{p} &= R^{q}(\boldsymbol{C}, \boldsymbol{x}_{0}) = (R_{s}^{q})_{l}^{l}(\boldsymbol{C}, \boldsymbol{x}_{0}) = \frac{|\boldsymbol{c}_{l}(\hat{\boldsymbol{x}} - \boldsymbol{x}_{0})|}{\|\hat{\boldsymbol{x}} - \boldsymbol{x}_{0}\|_{q}} \leq \frac{|\boldsymbol{c}_{l}(\boldsymbol{x}' - \boldsymbol{x}_{0})|}{\|\boldsymbol{x}' - \boldsymbol{x}_{0}\|_{q}} \\ \implies \quad \|\boldsymbol{c}_{l}(\delta) - \boldsymbol{c}_{l}\|_{p}\|\boldsymbol{x}' - \boldsymbol{x}_{0}\|_{q} \leq |\boldsymbol{c}_{l}(\boldsymbol{x}' - \boldsymbol{x}_{0})|, \\ \implies \quad |(\boldsymbol{c}_{l}(\delta) - \boldsymbol{c}_{l})(\boldsymbol{x}' - \boldsymbol{x}_{0})| \leq |\boldsymbol{c}_{l}(\boldsymbol{x}' - \boldsymbol{x}_{0})|, \\ \implies \quad |(\boldsymbol{c}_{l}(\delta)(\boldsymbol{x}' - \boldsymbol{x}_{0}) - \underbrace{\boldsymbol{c}_{l}(\boldsymbol{x}' - \boldsymbol{x}_{0})}_{<0}| \leq |\boldsymbol{c}_{l}(\boldsymbol{x}' - \boldsymbol{x}_{0})|. \end{aligned}$$

By the strict efficiency of x_0 with respect to the original problem, the inconsistency of the last inequality is reached.

Now, $\mathbf{x}_0 \in \mathbf{\Pi}(\mathbf{C}(\delta), \mathbf{X})$.

(2) To prove that x_0 is not strictly efficient for the perturbed problem, all of the objective vectors c_i are perturbed in the sensitive direction, this means that

$$c_{i1}(\delta) = c_{i1} + \operatorname{sign}(\hat{x}_1 - (x_0)_1)\delta_1$$

:

$$c_{in}(\delta) = c_{in} + \operatorname{sign}(\hat{x}_1 - (x_0)_1)\delta_n,$$
(14)

where $\delta_i \ge 0$ for all i = 1, ..., L. Also, $|\delta_l| = (R_s^q)_l^l$ and for all $i = 1, ..., L, i \ne l, |\delta_i| = (R_s^q)_l^l$. $\forall i = 1, ..., L$:

$$\begin{aligned} \|\boldsymbol{c}_{i}(\delta) - \boldsymbol{c}_{i}\|_{p} &= (R_{s}^{q})_{i}^{l}(\boldsymbol{C}, \boldsymbol{x}_{0}) = \frac{|\boldsymbol{c}_{i}(\hat{\boldsymbol{x}} - \boldsymbol{x}_{0})|}{\|\hat{\boldsymbol{x}} - \boldsymbol{x}_{0}\|_{q}}, \\ \implies \underbrace{\boldsymbol{c}_{i}(\delta)(\hat{\boldsymbol{x}} - \boldsymbol{x}_{0}) - \boldsymbol{c}_{i}(\hat{\boldsymbol{x}} - \boldsymbol{x}_{0})}_{\geq 0} = \underbrace{-\boldsymbol{c}_{i}(\hat{\boldsymbol{x}} - \boldsymbol{x}_{0})}_{\geq 0}, \\ \implies c_{i}(\delta)(\hat{\boldsymbol{x}} - \boldsymbol{x}_{0}) \geq 0. \end{aligned}$$

Then, $x_0 \notin S(C(\delta), X)$.

(3) To prove that \hat{x} is efficient for the perturbed problem, proof by contradiction is used. Assume that $\exists x' \neq \hat{x} \in X$ such that

$$\boldsymbol{c}_l(\delta)(\boldsymbol{x}' - \hat{\boldsymbol{x}}) \ge 0. \tag{15}$$

Then,

$$\boldsymbol{c}_l(\delta)(\boldsymbol{x}'-\boldsymbol{x}_0)+\boldsymbol{c}_l(\delta)(\boldsymbol{x}_0-\hat{\boldsymbol{x}})\geq 0.$$

 x_0 is efficient for the perturbed problem. Then,

$$\boldsymbol{c}_l(\delta)(\boldsymbol{x}'-\boldsymbol{x}_0) \leq 0 \Longrightarrow \boldsymbol{c}_l(\delta)(\boldsymbol{x}_0-\hat{\boldsymbol{x}}) \geq 0.$$

By (10), $c_l(\delta)(\hat{x} - x_0) \ge 0$ which means that $c_l(\delta)(x_0 - \hat{x}) = 0$. x_0 is strictly efficient for the original problem. Therefore, the equality (15) is false. Then $\hat{x} \in \Pi(C(\delta), X)$.

(4) The proof of the non-strict efficiency of \hat{x} is straightforward. In fact, in (2):

$$\boldsymbol{c}_i(\delta)(\hat{\boldsymbol{x}}-\boldsymbol{x}_0)\geq 0$$

is already proved. Then, $\hat{x} \notin S(C(\delta), X)$.

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PROPOSITION 4.3 Assume that $c_i(\hat{x} - x_0) \le 0$ for all i = 1, ..., L and that all the objective vectors c_i are perturbed in the sensitive direction and the perturbations are equal to the quasi-stability radius, except for at least one objective vector, c_k , which is perturbed slightly beyond the quasi-stability radius. Then x_0 becomes weakly efficient but not efficient for the perturbed problem and \hat{x} becomes efficient in its place.

Proof

- (1) The weak efficiency of x_0 is straightforward.
- (2) Proof by contradiction is used to prove that x_0 is not efficient for the perturbed problem. Assume that x_0 is efficient for the perturbed problem. Then, $\nexists x \neq x_0 \in X$, $\exists i_0 \in X$.
 - $\{1,\ldots,L\}, \quad \boldsymbol{c}_{i_0}(\delta)\boldsymbol{x} \geq \boldsymbol{c}_{i_0}(\delta)\boldsymbol{x}_0.$
 - If $i_0 \neq k$, then $c_k(\delta)(\hat{x} x_0) = 0$, which is inconsistent.
 - If $i_0 = k$, then

$$\exists \mathbf{x} \neq \mathbf{x}_0 \in \mathbf{X}, \quad \mathbf{c}_k(\delta)(\mathbf{x} - \mathbf{x}_0) \ge 0,$$

$$\Longrightarrow \quad \mathbf{c}_k(\delta)(\hat{\mathbf{x}} - \mathbf{x}_0) < 0,$$

$$\Longrightarrow \quad (\mathbf{c}_k(\delta) - \mathbf{c}_k)(\hat{\mathbf{x}} - \mathbf{x}_0) < -\mathbf{c}_k(\hat{\mathbf{x}} - \mathbf{x}_0).$$

All the perturbations are made in the sensitive direction. Then:

$$\|\boldsymbol{c}_k(\delta) - \boldsymbol{c}_k\|_p < (R_s^q)_k^l,$$

which is in contradiction with the assumption.

(3) Again, proof by contradiction is used to prove that \hat{x} is efficient. Assume that $\exists x' \neq \hat{x} \in X$ such that $c_k(\delta)(x' - \hat{x}) \ge 0$. Then,

$$\boldsymbol{c}_k(\delta)(\boldsymbol{x}'-\boldsymbol{x}_0)+\boldsymbol{c}_k(\delta)(\boldsymbol{x}_0-\hat{\boldsymbol{x}})\geq 0.$$

 x_0 is not efficient for the perturbed problem. Then,

$$\boldsymbol{c}_k(\delta)(\boldsymbol{x}'-\boldsymbol{x}_0) \leq 0 \Longrightarrow \boldsymbol{c}_k(\delta)(\boldsymbol{x}_0-\hat{\boldsymbol{x}}) \geq 0.$$

The last inequality is contradictory. Then $\hat{x} \in \Pi(C(\delta), X)$.

Remark 5 Use the same assumption as in Proposition 4.3 and consider that, for at least one *i*, $c_i(\hat{x} - x_0) > 0$. In that case only the entries of the row *k* for which $c_k(\hat{x} - x_0) \le 0$ are perturbed and the perturbations are equal to the quasi-stability radius. This means $||c_k(\delta) - c_k|| = (R_s^q)_k^l$ for all *k* for which $c_k(\hat{x} - x_0) \le 0$. Then x_0 becomes weakly efficient for the perturbed problem and \hat{x} becomes efficient in its place.

COROLLARY 4.4 Assume that all the objective vectors \mathbf{c}_i are perturbed slightly beyond the quasistability radius in the sensitive direction. Then \mathbf{x}_0 becomes inferior for the perturbed problem and $\hat{\mathbf{x}}$ becomes strictly efficient in its place.

Sketch of proof The proof follows the same principle as in the preceding propositions. Assuming that all the objective vectors \mathbf{c}_i are perturbed slightly beyond the quasi-stability radius in the sensitive direction means that for all i = 1, ..., L, for which $\mathbf{c}_i(\hat{\mathbf{x}} - \mathbf{x}_0) \le 0$, $\|\mathbf{c}_i(\delta) - \mathbf{c}_i\|_p > (R_s^q)_i^l$.

With the information presented above, the Summary 3.6 can be refined. The summary is introduced as a quick reference to the four points described above. All the details and assumptions made above are omitted for the sake of clarity.

SUMMARY 4.5

- (1) If \mathbf{x}_0 is a strictly efficient solution and the perturbations to the input matrix \mathbf{C} are less than the quasi-stability radius, \mathbf{x}_0 remains strictly efficient for the perturbed problem.
- (2) If all the objective vectors c_i are perturbed and the perturbations are equal to the quasistability radius, x_0 becomes efficient but not strictly efficient for the perturbed problem $(R^q(C(\delta), x_0) = 0).$
- (3) If all the objective vectors \mathbf{c}_i are perturbed and the perturbations are equal to the quasistability radius, except for at least one objective vector that is perturbed slightly beyond the quasi-stability radius, \mathbf{x}_0 becomes weakly efficient for the perturbed problem.
- (4) If all the objective vectors \mathbf{c}_i are perturbed slightly beyond the quasi-stability radius, \mathbf{x}_0 becomes inferior for the perturbed problem.

The Summary 4.5 shows that the transition from strictly efficient points to inferior points happens smoothly, in the same fashion in which the transition from optimality to inferiority goes through the alternative optima in the scalar optimization case. Summary 4.5 considers perturbations of the *C* matrix. Perturbations in the elements of *A* and *b* affect the feasibility of different points. For example, perturbing *A* and *b* in such a way that the feasible domain shrinks can cause the feasible strictly efficient point x_0 to end up on the boundary of the new feasible domain (its new slack is 0, therefore one of the new constraints is binding). If the perturbation is increased, x_0 may become infeasible. On the other hand, if *A* and *b* are perturbed in such a way that the feasible domain expands, then new points may become feasible and may dominate the original strictly efficient point x_0 . Therefore, the change of the strictly efficiency status, in this case, is just a consequence of the change in the number of feasible points.

4.2. Regularization

Finding the sensitive direction is particularly useful since any perturbation in the input data triplet C, A and b in the opposite direction of the sensitive one increases the stability radius/region as shown in Figure 6. This can constitute an approach to the regularization technique. One of the strengths of this regularization is that it can be applied in a step-wise fashion; this can be as simple as perturbing only one coefficient in the opposite direction as d. This allows the decision maker easily to understand the impact that the perturbation had on the stability radius as well as applying the perturbations in a preferred order. Such a sequential regularization approach can continue until the decision maker is comfortable working with the solution. The decision maker should note that the increase in the quasi-stability radius is always less than or equal to the strategic perturbation.

This regularization technique is useful to decision makers for several reasons:

- it is intuitive and uses only the information provided by the quasi-stability radius calculation,
- it allows the quasi-stability radius of quasi-stable solutions to be increased,
- it allows the quasi-unstable solutions to be transformed into quasi-stable solutions,
- it allows the justification of selecting a solution with an inadequate quasi-stability radius when expecting perturbations in the data triplet *C*, *A* and *b* to occur in the future,
- it allows a complex problem to be investigated and to determine what factors limit the problem's robustness.

4.3. Uncertainty analysis algorithm

The preceding results are used here to formulate a systematic approach to analysing linear integer single or multi-objective optimization problems with uncertainty in the input data triplet C, A and



Figure 7. The systematic decision making algorithm.

b. The algorithm is presented in Figure 7 and incorporates methodologies to search for preferred solutions, to quantify their robustness and, if necessary, to increase robustness. Specifically, the algorithm details the following.

- (1) The *formulation* of the multi-objective linear integer optimization problem. Input the data triplet C, A and b, the associated uncertainty and expected future changes. As well, the decision maker must enter the required scaling method and the weighting vectors to emphasize or de-emphasize objectives and constraints.
- (2) Soliciting the correct solution preferences from the decision maker.
- (3) Solving the optimization problem for an efficient solution x_0 .
- (4) Calculating the quasi-stability radius of x_0 .
- (5) Validating the quasi-stability radius against the input data uncertainty and expected future changes. If the radius accommodates the uncertainty and the expected changes have a positive impact, the results are presented to the user, who has the choice to continue the search or stop.

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If the radius is inadequate, the results are recorded and steps (3) to (5) are repeated with the next most preferred solution. If this process fails to find a satisfactory x_0 , the regularization technique is initiated.

(6) The *regularization technique* begins with selecting the most preferred solution from the recorded solution history. This is followed by calculating the perturbations required to make the selected solution quasi-stable enough to accommodate the input uncertainty. If the changes are realistic and can be implemented, the process stops, or else this step is repeated.

If the algorithm fails to reveal a satisfactory solution that accommodates the input uncertainty, the user should focus on reducing the highest levels of uncertainty before restarting the algorithm.

5. Case studies: the truck allocation problem

In this example, the problem is defined first, and then the set of efficient points and the respective quasi-stability radii are computed. For a selected strictly efficient point the sensitive directions are computed and, based on this information, the effect of different types of perturbation is analysed. The numerical results are in agreement with Summary 4.5 presented in Section 4.1. The proposed regularization technique, based on the computation of the sensitive direction, is also illustrated. In the numerical example, it is used to increase the quasi-stability radius of an already quasi-stable point. For this problem, the impact of uncertainty in one objective over the other is emphasized to demonstrate how this influences the decision makers' choice of efficient solutions.

This problem consists of three integer decision variables, three box constraints, one linear inequality constraint and two objective functions. The variables x_1 , x_2 and x_3 denote the number of 360, 600, 900 tonne trucks working during the shift, respectively. The objective functions and the constraints are:

$$\min_{x} \{x_1 + x_2 + x_3\} \quad \text{number of trucks,}$$

$$\min_{x} \{100x_1 + 200x_2 + 350x_3\} \quad \text{operating cost (dollars per hour),}$$

$$3100 \le 360x_1 + 600x_2 + 900x_3 \le 6180 \quad \text{ore production (tonnes per hour),}$$

$$0 \le x_1 \le 3,$$

$$0 \le x_2 \le 4,$$

$$0 \le x_3 \le 3.$$
(16)

Or

$$\max_{x} Cx$$
s.t. $Ax \le b$, (17)
 $x \ge 0$, $x \in \mathbb{Z}^{n}$,

where

$$\boldsymbol{C} = \begin{bmatrix} -1 & -1 & -1 \\ -100 & -200 & -350 \end{bmatrix}, \boldsymbol{A} = \begin{bmatrix} -360 & -600 & -900 \\ 360 & 600 & 900 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -3100 \\ 6180 \\ 3 \\ 4 \\ 3 \end{bmatrix}.$$
(18)

Table 2. Strictly efficient, efficient and weakly efficient solutions.

S(C, X)	$\Pi(C,X)\setminus S(C,X)$	$P(C,X) \setminus \Pi(C,X)$
(0,1,3), (2,1,2), (2,4,0)	N/A	(2,0,3), (1,2,2), (0,4,1), (3,2,1)

Table 2 presents the solutions of the problem: three strictly efficient points and four weakly efficient points. Thirty-three points are inferior.

First, the strictly efficient point $\mathbf{x}_0 = (2, 1, 2)$ will be analysed. Since the coefficients of the two objective functions differ by two orders of magnitude, the objectives must be scaled. The average value of each of the objective function coefficients was chosen for scaling. Therefore, using $\tilde{c}_i = \sum_{j=1}^N |c_{ij}|/N$, i = 1, ..., L, to scale the objective coefficients gives $\bar{c}_1 = (-1, -1, -1)$ and $\bar{c}_2 = (-0.46, -0.92, -1.62)$.

Calculating the quasi-stability radius with respect to criterion matrix \overline{C} using $p = \infty$, q = 1 (see Table 1) gives

$$\bar{R}^q_{\rm s}(\bar{C}, x_0) = 0.0384,$$

where $\hat{x} = (0, 4, 1)$ is the feasible point that becomes dominant to x_0 (see formula 11). The stability radius is limited by objective 2, therefore

$$(\bar{R}_{s}^{q})(\bar{C}, \mathbf{x}_{0}) = \frac{-\bar{c}_{2}(\hat{x} - \mathbf{x}_{0})}{\|\hat{x} - \mathbf{x}_{0}\|_{1}} = (\bar{R}_{s}^{q})_{2}^{2},$$

$$(\bar{R}_{s}^{q})_{1}^{2} = \frac{-\bar{c}_{1}(\hat{x} - \mathbf{x}_{0})}{\|\hat{x} - \mathbf{x}_{0}\|_{1}} = 0,$$

$$(R_{s}^{q})^{1} = (\bar{R}_{s}^{q}) * \tilde{c}_{1} = \frac{1}{26} = 0.0384,$$

$$(R_{s}^{q})^{2} = (\bar{R}_{s}^{q}) * \tilde{c}_{2} = \frac{325}{39} = 8.33,$$

$$\hat{x} - \mathbf{x}_{0} = (-2, 3, -1).$$

Table 3 is a summary of the possible perturbations to the original problem.

Now, suppose that the decision maker knows that the uncertainty in the second objective is ten times smaller than the uncertainty of the first objective. To take this information into account, the

Table 3. Effect of different types of perturbation.

Status of x_0	Conditions	Perturbations ($c_i(\delta) = c_i - \delta_i$)
$\overline{x_0 \in S(C(\delta), X)}$	$\begin{aligned} c_{1k}(\delta) - c_{1k} &< (R_{\rm s}^q)^1 \\ c_{2k}(\delta) - c_{2k} &< (R_{\rm s}^q)^2 \end{aligned}$	$\delta_1 : (-0.03, 0.03, -0.03) \\ \delta_2 : (-8, 8, -8)$
$x_0 \in \Pi(C(\delta), X)$	$ c_{1k}(\delta) - c_{1k} = (R_s^q)^1$	δ_1 : (0,0,0) δ_2 : (-8.3, 8.3, -8.3)
$\boldsymbol{x}_0 \in \boldsymbol{P}(\boldsymbol{C}(\delta), \boldsymbol{X})$	$ c_{1k}(\delta) - c_{1k} > (R_s^q)^1$	δ_1 : (0,0,0) δ_2 : (-8.5, 8.5, -8.5)
	$\begin{aligned} c_{1k}(\delta) - c_{1k} &> (R_{\rm s}^q)_1^1 \\ c_{2k}(\delta) - c_{2k} &> (R_{\rm s}^q)_2^1 \end{aligned}$	$\delta_1 : (-0.01, 0.01, -0.01) \\ \delta_2 : (-8.5, 8.5, -8.5)$
Regularization of $x_0 \in S(C(\delta), X)$	$\begin{aligned} c_{1k}(\delta) - c_{1k} &< (R_s^q)^1 \\ c_{2k}(\delta) - c_{2k} &< (R_s^q)^2 \\ \text{Opposite direction to} \\ \text{sensitive one} \end{aligned}$	$c_i(\delta) = c_i - \delta_i$ $\delta_1 : (0, 0, 0)$ $\delta_2 : (5, 0, 0)$ $\bar{R}_s^q(\boldsymbol{C}(\delta), \boldsymbol{x}_0) = 0.0465 > 0.038$

Strictly efficient point	No uncertainty		With uncertainty	
	$(R_s^q)^1$	$(R_s^q)^2$	$(R_s^q)^1$	$(R_s^q)^2$
(2,1,2)	0.038	8.3	0.2	N/A
(0,1,3)	0.2	N/A	0.2	N/A
(2,4,0)	0.058	12.5	0.577	12.5

Table 4. Robustness of the strictly efficient solutions with and without accounting for the varying degrees of uncertainty in the objectives.

impact of the second objective on the stability radius calculation is de-emphasized by using $\bar{c_2}^* = 10\bar{c_2}$. This yields $\bar{c_1} = (-1, -1, -1)$ and $\bar{c_2}^* = (-4.6, -9.2, -16.2)$. The dimensioned stability radii, with and without taking the uncertainty into account, are shown for all the strictly efficient points in Table 4. Notice that even when the uncertainty of the second objective is not accounted, the scaled variables are considered. Therefore, to return to the original variables, two different stability radii are obtained, one for each objective function. The cases marked as N/A in Table 4 occur when $c_i(x - x_0) > 0$. Here it can be seen that the most robust solution is $x_0 = (2, 4, 0)$ before and after accounting for the uncertainty of the first objective; however, the decision maker may have higher uncertainty in the coefficients of the first objective function than the allowable value of 0.058. If the uncertainty of the first objective had not been accounted for, he would have erroneously dismissed this solution or attempted to regularize a problem that does not require regularization. Previously, this point was unavailable to the decision maker because its quasistability radius was too small. This demonstrates the importance of incorporating the effect of varying degrees of uncertainty in the analysis.

In practice, the strictly efficient point $x_0 = (2, 1, 2)$ represents the number of tracks with different sizes that reaches a Pareto optimum of the problem (16). Based on this information, an optimal solution of the target calculation of the local quasi-stability radius is given. This solution gives a new strategy of truck allocation $\hat{x} = (0, 4, 1)$ that preserves efficiency with respect to the scaled and new objective functions. All the others possible perturbations in the objective functions that preserve efficiency are given in Table 3. This latest information can be used to increase the quasi-stability region ($\bar{R}_{s}^{q}(C(\delta), \mathbf{x}_{0}) = 0.0465 > 0.038$). Besides, with general information on the perturbations of the different objective functions (the uncertainty in the second objective is ten times smaller than the uncertainty of the first objective for instance), a robust Pareto optimal truck allocation $x_0 = (2, 4, 0)$, for which perturbation in one of the objective functions (the first one) does not lead to new calculation or regularization, is obtained. This latest is helpful for the decision maker in the sense that it gives a truck allocation that will be 'immunized' against uncertainty without additional calculation. To conclude, regularization may help to understand the impact of perturbation by making use of the local quasi-stability region and to work on the efficient point to get preferences order on solutions. For instance, which truck sizes and how many of them should be used to stay in the quasi-stability region? In the mean time, it helps to emphasize or de-emphasize perturbations on some objective functions, which is another way to deal with dimensional inconsistency and magnitude in the objective functions. Further issues and discussion on dimensional inconsistency and the magnitude of parameters and objective functions, leading to problems in scaling, are investigated in an upcoming article (Seck et al. 2012), which also includes a numerical example.

6. Summary and conclusions

This article presented a systematic approach to analysing a linear, integer, multi-objective optimization problem with uncertainty in the input data. The main contribution of the article

is the unification of a series of highly theoretical articles and the definition of the local quasistability radius. The importance of this local quasi-stability radius is emphasized when dealing with uncertain multi-objective linear optimization problems. Also an approach is introduced to compute the general local quasi-stability radius for simultaneous variations in all of the problem data. Furthermore, the article synthesizes a structured analysis approach to the problem of uncertain integer linear vector optimization problems, which was illustrated with case studies drawn from the mining industry.

One of the novel features of this article is that the concept of solution stability is used to address a number of important quantitative issues that decision makers must routinely face. These include: how robust a solution is to problem uncertainty; what the limiting level of uncertainty is to which the solution is robust; to which directions of variation in the input data is the solution most sensitive; and so forth. This work highlights the importance of theoretical stability analysis developments that have largely been available in Russian literature, and extends work dealing with a number of practical implementation issues.

The present work has raised a number of issues that remain open, including: dealing with scaling issues in the input data; emphasizing the uncertainty of some input data over others; and the extension of stability ideas to quadratic, and possibly nonlinear, programming problems. These issues are currently under active investigation.

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Appendix A. Quasi-stability radius with respect to the objective

A generalization of the quasi-stability radius formula (7) is

$$R^{q}(C, \mathbf{x}_{0}) = \min_{\substack{\mathbf{x} \neq \mathbf{x}_{0} \\ A\mathbf{x} = \mathbf{b} \leq 0}} \left(\frac{\|C(\mathbf{x} - \mathbf{x}_{0})\|_{p}}{\|\mathbf{x} - \mathbf{x}_{0}\|_{q}} \right).$$
(A1)

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The ∞ -norm (maximum) is used in the numerator of formula (7) to ensure agreement with the relation of order used throughout the article. The type of efficiency used (*i.e.* Pareto efficiency) is based on the component-wise relation of order, meaning that ' x_1 is better than x_2 ' (or x_1 improves x_2) is defined by $c_i x_1 \ge c_i x_2$ for all $i = 1, \ldots, L$. Choosing a different relation of order yields a different type of efficiency and a different stability radius formula. Now, if instead of using the component-wise relation of order, a relation of order in an average sense is defined, hence ' x_1 is better than x_2 ' if $||Cx_1||_1 \ge ||Cx_2||_1$, which is a different type of efficiency.

Using

$$\|C(x - x_0)\|_1 \le \|C\|_{\infty} \|x - x_0\|_1, \tag{A2}$$

which can be derived from the Hölder inequality for the case $p = \infty$, q = 1, a new formula for the quasi-stability radius is

$$R^{1}(C, \mathbf{x}_{0}) = \min_{\substack{\mathbf{x} \neq \mathbf{x}_{0} \\ A\mathbf{x} - \mathbf{b} \leq 0}} \left(\frac{\|C(\mathbf{x} - \mathbf{x}_{0})\|_{1}}{\|\mathbf{x} - \mathbf{x}_{0}\|_{1}} \right)$$
$$= \min_{\substack{\mathbf{x} \neq \mathbf{x}_{0} \\ A\mathbf{x} - \mathbf{b} \leq 0}} \left(\frac{\sum_{i=1}^{L} \left| \sum_{j=1}^{n} c_{ij}(x_{j} - x_{0j}) \right|}{\sum_{k=1}^{n} |x_{k} - x_{0k}|} \right).$$
(A3)

The perturbations allowed by this stability radius are $\|C(\delta) - C\|_1 < R^1$. Using the definition of the 1-norm when dealing with matrices,

$$\max_{i=1,\dots,n}\sum_{i=1}^{L}|c_{ij}(\delta)-c_{ij}|<\delta.$$

Therefore, the entries of every column j = 1, ..., n s.t. $\sum_{i=1}^{L} |c_{ij}(\delta) - c_{ij}| < R^q$ can be perturbed, meaning that the sum of absolute values of the perturbations in one column must be less than the stability radius.

This shows that the formula (7) of the stability radius can be easily generalized to (A3). An even more general formula would be (A1).

Appendix B. Quasi-stability radius (input matrix A and vector b)

The goal of this section is to provide the decision maker with stability information about the selected efficient solution with respect to uncertainty in the input matrix A and input vector b. Here, the decision maker is primarily concerned with knowing whether the efficient solution x_0 from problem (1) remains efficient in the perturbed problem (B1). This is resolved by the quasi-stability radius which quantifies the limiting level of perturbations such that the efficiency and feasibility of the solution are preserved.

$$\max_{x} Cx$$

s.t. $A(\delta)x \le b(\delta),$ (B1)
 $x \in \mathbb{Z}^{n}.$

Assume that only the entries of the matrix A are perturbed. The distance between problem (1) and the perturbed one (B1) is $\max_{1 \le i \le m} \|A_i(\delta) - A_i\|_p$, where $\|\cdot\|_p$ stands for the *p*-norm on \mathbb{R}^n .

The following theorem gives the formula for the local quasi-stability radius (see Definition 3.3) with respect to matrix C for an efficient solution x_0 .

THEOREM B.1 Let \mathbf{x}_0 be an efficient solution of problem (1) and $1 \le p \le +\infty$. Then, if only additive perturbations with respect to the constraints matrix \mathbf{A} are considered, the quasi-stability radius on x_0 is given by

$$R^{q}(A, \mathbf{x}_{0}) = \min\{(R^{q})_{1}(A, \mathbf{x}_{0}), (R^{q})_{2}(A, \mathbf{x}_{0})\},$$
(B2)

where

$$(R^{q})_{1}(\boldsymbol{A},\boldsymbol{x}_{0}) = \min_{\substack{\boldsymbol{x}\neq\boldsymbol{x}_{0}\\C(\boldsymbol{x}-\boldsymbol{x}_{0})>0}} \max_{\substack{1\leq i\leq m\\c(\boldsymbol{x}-\boldsymbol{x}_{0})>0\\c(\boldsymbol{x}-\boldsymbol{x}_{0})>b_{i}}} \frac{|\boldsymbol{A}_{i}\boldsymbol{x}-b_{i}|}{\|\boldsymbol{x}\|_{q}}$$
(B3)

and

$$(R^{q})_{2}(\boldsymbol{A},\boldsymbol{x}_{0}) = \min_{1 \le i \le m} \frac{|\boldsymbol{A}_{i}\boldsymbol{x}_{0} - \boldsymbol{b}_{i}|}{\|\boldsymbol{x}_{0}\|_{q}}.$$
 (B4)

Sketch of proof In deriving the quasi-stability with respect to the input matrix A, perturbations to A can cause an infeasible point x that is dominant to x_0 to become feasible, or cause x_0 to become infeasible.

Therefore, regarding the former case, the critical perturbations as given by $a_i(\delta)x - b_i = 0$ are only calculated for constraints i where $a_i x - b_i > 0$ and for alternative solutions x that dominate x_0 , i.e. $C(x - x_0) \ge 0$. Following a similar logic to that used in the previous section, the quasi-stability radius with respect to the matrix A while increasing the feasible domain can be computed by following Equation (B3) (a similar proof for global quasi-stability can be found in Leontev and Mamutov 1988 and Emelichev and Krichko 1999).

With regard to the latter case, the critical perturbations are given by $a_i(\delta)x_0 - b_i = 0$, and the stability radius equation with respect to the matrix A while decreasing the feasible space is given by Equation (B4).

To guarantee that the efficiency and feasibility of x_0 are preserved, the final stability radius equation with respect to the matrix A is given by Equation (B2), where the perturbations to matrix A must satisfy the condition

$$\|\boldsymbol{a}_i(\delta) - \boldsymbol{a}_i\|_p < R^q(\boldsymbol{A}, \boldsymbol{x}_0).$$
(B5)

The line of reasoning behind deriving the quasi-stability radii with respect to the vector **b**, the matrix **A** and combinations thereof is very similar. Therefore, the next theorems are presented without any proof or additional commentaries. Again, similar proofs concerned with the global quasi-stability radius can be found in Leontev and Mamutov (1988) and Emelichev and Krichko (1999).

THEOREM B.2 Let x_0 be an efficient solution of problem (1) and $1 \le p \le +\infty$. Then, if only additive perturbations with respect to the constraints vector **b** are considered, the quasi-stability radius on x_0 is given by

$$R^{q}(\boldsymbol{b}, \boldsymbol{x}_{0}) = \min\{(R^{q})_{1}(\boldsymbol{b}, \boldsymbol{x}_{0}), (R^{q})_{2}(\boldsymbol{b}, \boldsymbol{x}_{0})\},$$
(B6)

where

$$(\mathbf{R}^{q})_{1}(\mathbf{b}, \mathbf{x}_{0}) = \min_{\substack{\mathbf{x}\neq\mathbf{x}_{0}\\C(\mathbf{x}-\mathbf{x}_{0})\geq 0}} \max_{\substack{\mathbf{x}\neq\mathbf{x}_{0}\\\mathbf{x}_{i}\neq\mathbf{x}_{b}}} \max_{\substack{\mathbf{a}\mid\mathbf{x}=\mathbf{b}_{i}\\\mathbf{x}_{i}\neq\mathbf{x}_{i}}} |\mathbf{A}_{i}\mathbf{x}-\mathbf{b}_{i}|$$
(B7)

and

$$(R^{q})_{2}(\boldsymbol{b},\boldsymbol{x}_{0}) = \min_{1 \le i \le m} |A_{i}\boldsymbol{x}_{0} - b_{i}|.$$
(B8)

THEOREM B.3 Let x_0 be an efficient solution of problem (1) and $1 \le p \le +\infty$. Then, if additive perturbations with respect to both the constraints matrix **A** and the constraints vector **b** are considered, the quasi-stability radius on x_0 is given by

$$R^{q}(\mathbf{A}, \mathbf{b}, \mathbf{x}_{0}) = \min\{(R^{q})_{1}(\mathbf{A}, \mathbf{b}, \mathbf{x}_{0}), (R^{q})_{2}(\mathbf{A}, \mathbf{b}, \mathbf{x}_{0})\},$$
(B9)

where

$$(R^{q})_{1}(A, \boldsymbol{b}, \boldsymbol{x}_{0}) = \min_{\substack{\boldsymbol{x} \neq \boldsymbol{x}_{0} \\ C(\boldsymbol{x} - \boldsymbol{x}_{0}) \ge 0}} \max_{\substack{1 \le i \le m \\ A_{i}\boldsymbol{x} > b_{i}}} \frac{|A_{i}\boldsymbol{x} - b_{i}|}{\|\boldsymbol{x}\|_{q} + 1}$$
(B10)

and

$$(R^{q})_{2}(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{x}_{0}) = \min_{1 \le i \le m} \frac{|\boldsymbol{A}_{i}\boldsymbol{x}_{0} - \boldsymbol{b}_{i}|}{\|\boldsymbol{x}_{0}\|_{q} + 1}.$$
(B11)

. .

Remark B6 If some of the constraints are known with certainty, meaning that some of the rows of the matrix A – denoted by A_i and the corresponding right-hand side entries of the vector **b** – denoted by b_i contain no uncertainty, then the quasi-stability radius with respect to the constraints have to be carefully modified. For instance, (B3) becomes

$$(R^{q})_{1}(A, \mathbf{x}_{0}) = \min_{\substack{\mathbf{x} \neq \mathbf{x}_{0} \\ C(\mathbf{x} - \mathbf{x}_{0}) \geq 0 \\ A_{i}\mathbf{x} < b_{i} \text{(fixed constraints)}}} \max_{\substack{1 \le i \le m \\ A_{i}\mathbf{x} > b_{i} \\ i \neq j}} \frac{|A_{i}\mathbf{x} - b_{i}|}{\|\mathbf{x}\|_{q}}.$$
(B12)

Appendix C. Quasi-stability radius (matrices C, A and the vector b)

In many common situations, model parameters can appear in both C and A. In these cases, the effects of perturbations in the matrices cannot be analysed independently. The goal of this section is to provide the decision maker with stability information about the selected efficient solution with respect to uncertainty in the input data triplet C, A and b. Here, the primary concern is to know whether the efficient solution x_0 from problem (1) is still efficient in the perturbed problem (3). Again, this is resolved by the quasi-stability radius which quantifies the limiting level of perturbations such that efficiency and feasibility of the solution are preserved.

Following the same logic as in the previous sections, the local quasi-stability radius can be computed through the following theorem.

THEOREM B.4 Let \mathbf{x}_0 be an efficient solution of problem (1) and $1 \le p \le +\infty$. Then, if additive perturbations with respect to all of the objective matrix \mathbf{C} , the constraints matrix \mathbf{A} and the constraints vector \mathbf{b} are considered, the quasi-stability radius on x_0 is given by

$$R^{q}(C, A, b, x_{0}) = \min\{(R^{q})_{2}(A, b, x_{0}), (R^{q})_{1}(C, A, b, x_{0}), (R^{q})(C, x_{0})\},$$
(C1)

where

$$(R^{q})_{1}(C, A, b, x_{0}) = \min_{\substack{x \neq x_{0} \\ c_{l}(x - x_{0}) \leq 0 \\ A_{l} \neq > b_{l}}} \max_{\substack{1 \leq l \leq L \\ c_{l}(x - x_{0}) \leq 0 \\ A_{l} \neq > b_{l}}} \left\{ \frac{|A_{i}x - b_{i}|}{\|x\| + 1}, \frac{|c_{l}(x - x_{0})|}{\|x - x_{0}\|} \right\}.$$
(C2)

Sketch of proof In deriving the quasi-stability with respect to the input data triplet C, A and b, the efficiency of x_0 must be preserved with respect to

- any alternative solution lying inside the feasible domain Equation (7),
- any alternative solution lying outside the feasible domain Equation (C1), and
- perturbations that cause infeasibility Equation (C2) (it gives the slack of x_0).

Remark B7 This quasi-stability radius formula (C1), also given in Emelichev and Krichko (1999), is based on the implicit assumption that a model parameter can only appear in one of C, A or b; however, whenever C and A contain common model parameters, Equation (7) must be replaced by Equation (C3), which allows simultaneous variations of C, A and b. This modification yields a new formula, (C4), which represents the most general analytical expression of the quasi-stability radius:

$$R_{1}^{q}(\boldsymbol{C},\boldsymbol{x}_{0}) = \begin{cases} R^{q}(\boldsymbol{C},\boldsymbol{x}_{0}) & \text{if } \max_{\substack{1 \le i \le L \\ c_{i}(\boldsymbol{x}-\boldsymbol{x}_{0}) \le 0 \\ 0 & \text{elsewhere.} \end{cases}} \frac{|\boldsymbol{c}_{i}(\boldsymbol{x}-\boldsymbol{x}_{0})|}{\|\boldsymbol{x}-\boldsymbol{x}_{0}\|_{q}} > \max_{1 \le i \le m} \frac{|\boldsymbol{A}_{i}\boldsymbol{x}-\boldsymbol{b}_{i}|}{\|\boldsymbol{x}\|_{q}+1} \tag{C3}$$

$$R^{q}(C, A, b, x_{0}) = \min\{(R^{q})_{2}(A, b, x_{0}), (R^{q})_{1}(C, A, b, x_{0}), R^{q}_{1}(C, x_{0})\}.$$
(C4)

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